Rationality questions for Fano schemes of intersections of two quadrics

(joint work with Lena Ji)

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k: arbitrary field of characteristic $\neq 2$
$X \subset \mathbb{P}^N$: smooth complete intersection of 2 quadrics

$r \geq 0$, $F_r(X) :=$ Fano scheme of r-planes (i.e., $\mathbb{P}^r \subset \mathbb{P}^N$) on X.

Description of $F_r(X)$:

r	N odd	N even
$r = \lfloor \frac{N}{2} \rfloor - 1$ (max)	torsor under an abelian variety (Weil '50s)	finite, not geometrically integral
$0 \leq r \leq \lfloor \frac{N}{2} \rfloor - 2$	Fano, i.e., $-K$ ample	Fano, i.e., $-K$ ample

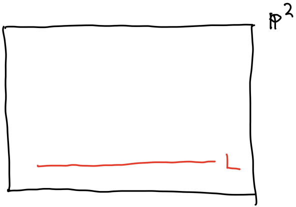
N odd: $F_r(X)$ is geometrically a certain moduli of vector bundles on a hyperelliptic curve (Desale–Ramanan '78, Ramanan '81).

$\rightarrow$ arithmetic applications

A variety is **rational** if it is birational to a projective space.

If there exists a line $L \subset X$ defined over k, consider the projection away from L: $\pi_L \colon X \dashrightarrow \mathbb{P}^{N-2}$.

Fibers of π_L?



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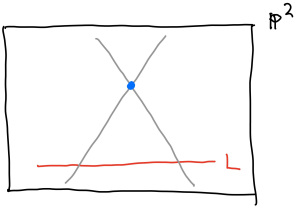

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Theorem 1 (Ji–S., '24)

If $F_{r+1}(X)(k) \neq \emptyset$, then $F_r(X)$ is rational.

Immediate consequence:

Corollary

$F_r(X)$ is geometrically rational for all $0 \leq r \leq \lfloor \frac{N}{2} \rfloor - 2$.

- $r = 0, \lfloor \frac{N}{2} \rfloor - 2$: known (latter by combining: Desale–Ramanan '77, Newstead '80, Bauer '91, Casagrande '15)
- $0 < r < \lfloor \frac{N}{2} \rfloor - 2$: new, even over $\mathbb{C}$!

Theorem 2 (Ji–S., '24)

For $N \geq 6$, the following are equivalent:

1. $F_1(X)$ is separably unirational;
2. $F_1(X)$ is unirational;
3. $F_1(X)(k) \neq \emptyset$.

If $k = \mathbb{R}$, the above result holds for $F_r(X)$ for all $0 \leq r \leq \lfloor \frac{N}{2} \rfloor - 2$.

This extends an analogous result for $F_0(X) = X$ (Manin '86, Knecht '15, Colliot-Thélène–Sansuc–Swinnerton-Dyer '87, Benoist–Wittenberg '23).

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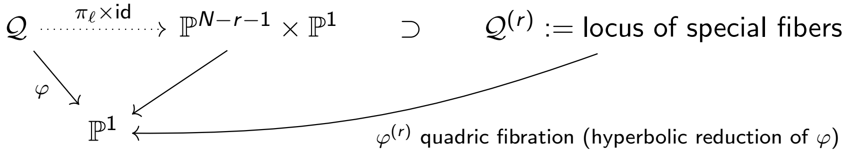
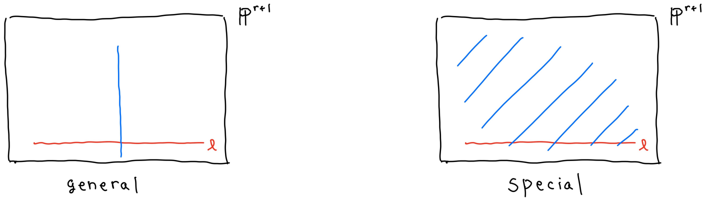
Let $\varphi \colon \mathcal{Q} \to \mathbb{P}^1$ be the pencil of quadrics, associated to X.

Assume $F_r(X)(k) \neq \emptyset$ and fix $\ell \in F_r(X)(k)$.



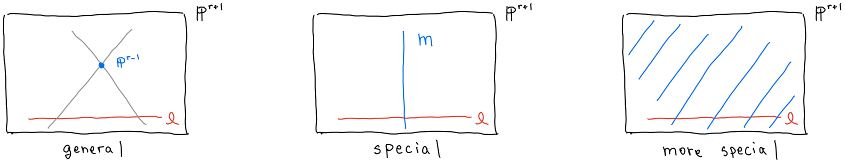
Fibers of $\pi_\ell \times \mathrm{id}$:



Similarly,

$$X \overset{\pi_\ell}{\dashrightarrow} \mathbb{P}^{N-r-1} \supset \widetilde{\mathcal{Q}}^{(r)} := \text{locus of special (and more special) fibers}$$

Fibers of π_ℓ:



Note

$$\widetilde{\mathcal{Q}}^{(r)} \overset{\sim}{\dashrightarrow} \mathcal{Q}^{(r)}, \; m \mapsto \langle \ell, m \rangle,$$

where the inverse is given by $\mathbb{P}^{N-r-1} \times \mathbb{P}^1 \to \mathbb{P}^{N-r-1}$.

The birational equivalence class of $\mathcal{Q}^{(r)}$ does NOT depend on ℓ.

Indeed,

$$\mathcal{Q}_{k(\mathbb{P}^1)} \simeq \mathcal{Q}^{(r)}_{k(\mathbb{P}^1)} \perp (\text{hyperbolic space})$$

as quadratic spaces, hence the Witt cancellation theorem shows that the isomorphism class of $\mathcal{Q}^{(r)}_{k(\mathbb{P}^1)}$ does not depend on ℓ.

Here is a birational structure theorem of $F_r(X)$ in terms of $\mathcal{Q}^{(r)}$.

Theorem 3 (Ji–S., '24)

One of the following conditions holds:

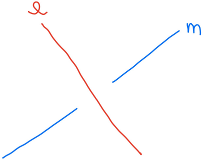
1. $F_r(X)$ is birational to $\mathrm{Sym}^{r+1}\,\mathcal{Q}^{(r)}$;
2. N is even and $r = \lfloor \frac{N}{2} \rfloor - 1$, in which case $F_r(X)$ is finite and not geometrically integral.

Two special cases were previously known before:

- $r = 0$, which claims $X \sim \mathcal{Q}^{(0)}$ (Colliot-Thélène–Sansuc–Swinnerton-Dyer '87);
- N is odd, $r = \lfloor \frac{N}{2} \rfloor - 1$, and $k = \overline{k}$ (Reid '72).

Proof of Thm 3:

$r = 1$: Let $m \in F_1(X)$ be general. Then $\langle \ell, m \rangle = \mathbb{P}^3$.



$$\langle \ell, m \rangle \cap X = \ell \cup m \cup m_1 \cup m_2$$

Define $F_1(X) \dashrightarrow \mathrm{Sym}^2 \mathcal{Q}^{(1)}$, $m \mapsto (m_1, m_2)$, which is generically one-to-one onto its image. Similar for $r > 1$. (Use a lemma of Reid.)

Finally, $\dim F_r(X) = \dim \mathrm{Sym}^{r+1} Q^{(r)} = (r+1)(N - 2r - 2)$, hence the above map is dominant, thus birational. Q.E.D.

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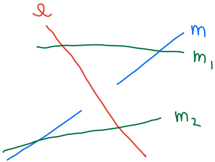

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Thm 3 $\Rightarrow$ Thm 1:

If $F_{r+1}(X)(k) \neq \emptyset$, then $\varphi^{(r)} \colon \mathcal{Q}^{(r)} \to \mathbb{P}^1$ has a section.
$\Rightarrow \mathcal{Q}^{(r)}$ is rational.
$\Rightarrow F_r(X) \sim \mathrm{Sym}^{r+1} Q^{(r)}$ is rational. Q.E.D.

We have used:
A symmetric power of a rational variety is rational (Mattuck '69).

Thm 3 $\Rightarrow$ Thm 2 (k arbitrary):

W.T.S. $\forall N \geq 6$, $F_1(X)(k) \neq \emptyset \Rightarrow F_1(X)$ separably uniratinonal.

A symmetric power of a separably unirational variety is separably unirational.

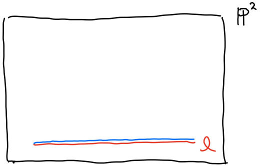
E.T.S. $\forall N \geq 6$, $F_1(X)(k) \neq \emptyset \Rightarrow \mathcal{Q}^{(1)}$ separably uniratinonal.

We prove this by induction on N.

<u>$N = 6$</u>: $X \subset \mathbb{P}^6$

$\varphi^{(1)} \colon \mathcal{Q}^{(1)} \to \mathbb{P}^1$ is a conic bundle with 7 singular fibers.

Moreover, $\mathcal{Q}^{(1)}(k) \neq \emptyset$,
because $\cap_{p \in \ell} T_p X = \mathbb{P}^2 \supset \ell$ and $(\cap_{p \in \ell} T_p X) \cap X = \ell$.



Such a conic bundle has a dominant map from $\mathbb{P}^2$ of degree 8 (Kollár–Mella '17).

$\therefore$ $\mathcal{Q}^{(1)}$ is separably unirational. (Recall $\operatorname{char} k \neq 2$.)

$\underline{N > 6}$: $X \subset \mathbb{P}^N$

Choose a general pencil of hyperplane sections of X containing ℓ.

We get $\mathcal{Q}^{(1)} \dashrightarrow \mathbb{P}^1$ whose generic fiber equals the hyperbolic reduction of the pencil associated to $Y \subset \mathbb{P}^{N-1}$ with respect to ℓ.

By the induction hypothesis, the generic fiber is separably unirational, and so is $\mathcal{Q}^{(1)}$. Q.E.D.

Thm 3 $\Rightarrow$ Thm 2 ($k = \mathbb{R}$):

$\widetilde{\mathcal{Q}}^{(r)} \subset \mathbb{P}^{N-r-1}$ has odd degree.
(For instance, $\widetilde{\mathcal{Q}}^{(0)} \subset \mathbb{P}^{N-1}$ is a cubic hypersurface.)
$\Rightarrow \mathcal{Q}^{(r)}$ has a 0-cycle of degree 1.
$\Rightarrow \mathcal{Q}^{(r)}(\mathbb{R}) \neq \emptyset$.

Apply a unirationality result (Kollár '99) to the quadric fibration $\varphi^{(r)} \colon \mathcal{Q}^{(r)} \to \mathbb{P}^1$. Q.E.D.

A conic bundle over $\mathbb{P}^1$ with a 0-cycle of degree 1 does not necessarily have a k-point (Colliot-Thélène–Coray '79).

Next: We will further analyze rationality of $F_r(X)$ for $r = \lfloor \frac{N}{2} \rfloor - 2$, the second maximal case.

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$N := 2g + 1$ ($g \geq 2$)

max $= g - 1$, second maximal $= g - 2$

Theorem 4 (Ji–S., '24)

Let $X \subset \mathbb{P}^{2g+1}$. Then:
$F_{g-2}(X)(k) \neq \emptyset$ and $\mathcal{Q}^{(g-2)}$ rational $\Leftrightarrow F_{g-1}(X)(k) \neq \emptyset$.

- $g = 2$: $X \subset \mathbb{P}^5$ is rational $\Leftrightarrow F_1(X)(k) \neq \emptyset$ (Hassett–Tschinkel 18' for $k = \mathbb{R}$, Benoist–Wittenberg '23 for k arbitrary).
- $g \geq 2$: partial converse to Thm 1, different from the full converse by a symmetric power: $\mathcal{Q}^{(g-2)} \leftrightarrow F_{g-2}(X) \sim \mathrm{Sym}^{g-1}\, \mathcal{Q}^{(g-2)}$.
- An analogue may fail for N even and $k = \mathbb{R}$.

Towards the proof of Thm 4:

$F_{g-1}(X)$ is a torsor under the Jacobian of C, where C is a hyperelliptic curve of genus g associated to $\varphi \colon \mathcal{Q} \to \mathbb{P}^1$ (Wang '18).

$$\begin{array}{ccc} F_g(\varphi) & \longrightarrow & \mathbb{P}^1 \\ & \searrow & \nearrow {\scriptstyle 2:1} \\ & C. & \end{array}$$

W.T.S. $F_{g-1}(X)$ splits $\Leftrightarrow$ $\mathcal{Q}^{(g-2)}$ defined & rational.

Note: $\dim \mathcal{Q}^{(g-2)} = 3$.

Idea: Clemens–Griffiths method à la Benoist–Wittenberg.

The goal is to show that $F_{g-1}(X)$ is a torsor under the intermediate Jacobian of $\mathcal{Q}^{(g-2)}$ ($\cong \mathrm{Jac}(C)$ as p.p.a.v.) and it splits when $\mathcal{Q}^{(g-2)}$ is rational.

This involves analysis on the algebraic equivalence class of a section of the quadric surface fibration $\varphi^{(g-2)} \colon \mathcal{Q}^{(g-2)} \to \mathbb{P}^1$.

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$N := 2g \ (g \geq 2)$

max $= g - 1$, second maximal $= g - 2$

Theorem 5 (Ji–S., 24')

Let $X \subset \mathbb{P}^{2g}$ over $\mathbb{R}$. Then:
$F_{g-2}(X)$ rational $\Leftrightarrow$ $F_{g-2}(X)(\mathbb{R})$ non-empty and connected.
Moreover, these conditions are equivalent to rationality of $\mathcal{Q}^{(g-2)}$.

- $\Rightarrow$ is true for all smooth projective varieties over $\mathbb{R}$ (Comessatti, 1912).
- $X \subset \mathbb{P}^6_{\mathbb{R}}$ rational $\Leftrightarrow$ $X(\mathbb{R})$ non-empty and connected (Hassett–Kollár–Tschinkel '22).
- An analogue may fail for N odd.

Towards the proof of Thm 5:

Let X as in Thm 5 and assume $F_{g-2}(X)(\mathbb{R}) \neq \emptyset$.

$\varphi^{(g-2)} \colon \mathcal{Q}^{(g-2)} \to \mathbb{P}^1$ is a conic bundle, hence $\mathcal{Q}^{(g-2)}$ is a geometrically rational surface.

Towards the proof of Thm 5:

Let X as in Thm 5 and assume $F_{g-2}(X)(\mathbb{R}) \neq \emptyset$.

$\varphi^{(g-2)}\colon \mathcal{Q}^{(g-2)} \to \mathbb{P}^1$ is a conic bundle, hence $\mathcal{Q}^{(g-2)}$ is a geometrically rational surface.

A geometrically rational surface defined over $\mathbb{R}$ is rational if and only if its real locus is non-empty and connected (Comessatti 1913).

$$\begin{array}{ccc}
\mathcal{Q}^{(g-2)} \text{ rational} & \overset{\text{Comessatti}}{\Longleftrightarrow} & \mathcal{Q}^{(g-2)}(\mathbb{R}) \text{ non-empty and connected} \\
\text{Thm 3 + Mattuck}\Big\Downarrow & & \Big\Uparrow \\
F_{g-2}(X) \text{ rational} & \underset{\text{Comessatti}}{\Longrightarrow} & F_{g-2}(X)(\mathbb{R}) \text{ non-empty and connected,}
\end{array}$$

where the right vertical arrow follows by studying the image of

$$\mathrm{Sym}^{g-1}\,\mathcal{Q}^{(g-2)}(\mathbb{R}) \to \mathrm{Sym}^{g-1}\,\mathbb{P}^1(\mathbb{R}) \xrightarrow{\sim} \mathbb{P}^{g-1}(\mathbb{R}).$$

Thank you!